

Strong twin events in mixed-state entanglement

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Abstract. Continuing the study of mixed-state entanglement in terms of opposite-subsystem observables the measurement of one of which amounts to the same as that of the other (so-called twins), begun in a recent article, so-called strong twin events, which imply biorthogonal mixing of states, are defined and studied. It is shown that for each mixed state there exists a Schmidt canonical (super state vector) expansion in terms of Hermitian operators, and that it can be the continuation of the mentioned biorthogonal mixing due to strong twins. The case of weak twins and nonhermitian Schmidt canonical expansion is also investigated. A necessary and sufficient condition for the existence of nontrivial twins for separable states is derived.

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1. Introduction

It was argued in a recent article [1] that the study of entanglement through twin observables, or shortly *twins*, is important for quantum communication and quantum information theories because it reveals very basic properties. Twin observables are opposite-subsystem observables such that the (subsystem) measurement of one of them amounts to a measurement also of the other. Equivalently put, the subsystem measurement of a twin gives rise, on account of entanglement, to an orthogonal state decomposition of the state of the opposite subsystem.

When a general, i. e., mixed or pure, composite-system state (statistical operator) ρ_{12} is given, twins (A_1, A_2) are algebraically defined as Hermitian (opposite subsystem) operators satisfying

$$A_1 \rho_{12} = A_2 \rho_{12}, \quad (1)$$

where A_1 is actually $A_1 \otimes I_2$, I_2 being the identity operator for the second subsystem, etc. It was shown [1] that (1) implies

$$[A_1, \rho_1] = 0, \quad [A_2, \rho_2] = 0, \quad (2a, b)$$

where $\rho_1 \equiv \text{Tr}_2 \rho_{12}$ and ρ_2 (defined symmetrically) are the subsystem states (the reduced statistical operators). The symbols Tr_i , $i = 1, 2$, denote the partial traces. Further, the so-called *detectable parts* A'_i of the twins, the restrictions of A_i to the ranges $\mathcal{R}(\rho_i)$, $i = 1, 2$, have equal and necessarily purely discrete spectra (but with possibly different multiplicities of the characteristic values except in the pure-state case). Further, the characteristic events (projectors) corresponding to the same characteristic value are twins.

Let (P_1, P_2) be a pair of nontrivial twin events (twin projectors) for ρ_{12} . Then we can decompose the statistical operator:

$$\rho_{12} = P_1 \rho_{12} + P_1^\perp \rho_{12}, \quad (3)$$

where P_1^\perp is the orthocomplementary projector of P_1 . In general, the terms on the RHS are not even Hermitian. First, we are going to investigate the more important case when (3) is a mixture of states.

2. Strong twin projectors and biorthogonal mixtures

Let (P_1, P_2) be a pair of nontrivial twin projectors for a composite-system statistical operator ρ_{12} .

Remark. Evidently, either both terms on the RHS of (3) are Hermitian or none of them. They are *Hermitian* if and only if the projector P_1 (or equivalently, P_1^\perp) *commutes* with ρ_{12} :

$$[P_i, \rho_{12}] = 0, \quad i = 1, 2, \quad (4)$$

(any one of the equalities implies the other), as seen by adjoining the terms in (3).

Hermiticity of the terms in (3) implies that they are statistical operators (up to normalization constants), i. e., that (3) is a *mixture*. Namely, if (4) is valid, then idempotency leads to $P_1\rho_{12} = P_1\rho_{12}P_1$, which is evidently a positive operator. Since

$$\text{Tr}P_1\rho_{12}P_1 \leq \text{Tr}\rho_{12} = 1,$$

the operator has a finite trace.

Definition 1. We call nontrivial twin events (projectors) either *strong twin events* (projectors), if they satisfy (4), or *weak twin events* (projectors) if (4) is not satisfied.

A strong twin event P_1 implies a mixture (3) of states that have a strong property called biorthogonality. To understand it, we first remind of (ordinary) orthogonality of states.

If ρ' and ρ'' are statistical operators with Q' and Q'' as their respective range projectors, then one has the known equivalences:

$$\rho'\rho'' = 0 \quad \Leftrightarrow \quad Q'Q'' = 0 \quad \Leftrightarrow \quad \mathcal{R}(\rho') \perp \mathcal{R}(\rho''), \quad (5)$$

where the last relation expresses orthogonality of the ranges.

Any of the three relations in (5) defines *orthogonality of states*.

Definition 2. If

$$\rho_{12} = w\rho'_{12} + (1-w)\rho''_{12}, \quad 0 < w < 1, \quad (6)$$

is a mixture of states such that

$$\rho'_i\rho''_i = 0, \quad i = 1, 2, \quad (7)$$

where $\rho'_1 \equiv \text{Tr}_2\rho'_{12}$ etc. are the reduced statistical operators, then we say that (6) is a *biorthogonal mixture*.

To prove a close connection between strong twin events and biorthogonal mixtures, we need another known general property of composite-system statistical operators ρ_{12} :

$$\rho_{12} = Q_1\rho_{12} = \rho_{12}Q_1 = Q_2\rho_{12} = \rho_{12}Q_2, \quad (8)$$

where Q_i is the range projector of the corresponding reduced statistical operator ρ_i , $i = 1, 2$.

Theorem 1. If P_1 is a nontrivial twin event, (3) is a *biorthogonal mixture* if and only if P_1 is a *strong twin event*.

Proof. Sufficiency. If P_1 is a strong twin projector and (6) is obtained by rewriting (3), then $w\rho'_{12} = P_1\rho'_{12}$ is valid, and this implies $\rho'_1 = P_1\rho'_1$ for the reduced statistical

operator, and, adjoining this, one arrives at $\rho'_1 = \rho'_1 P_1$. On the other hand, one has analogously $\rho''_{12} = P_1^\perp \rho''_{12}$ implying $\rho'_1 = P_1^\perp \rho'_1$. Finally,

$$\rho'_1 \rho''_1 = (\rho'_1 P_1)(P_1^\perp \rho''_1) = 0.$$

The symmetrical argument holds for the second tensor factor.

Necessity. If (6) is a biorthogonal mixture, then we define $P_i \equiv Q'_i$, $i = 1, 2$, i. e., we take the range projectors of the reduced statistical operators of ρ'_{12} as candidates for our twin projectors. On account of (8), we can write (6) as follows:

$$\rho_{12} = w Q'_1 Q'_2 \rho'_{12} Q'_1 Q'_2 + (1 - w) Q''_1 Q''_2 \rho''_{12} Q''_1 Q''_2.$$

Since in view of (5) biorthogonality (7) implies $Q'_i Q''_i = 0$, $i = 1, 2$, it is now obvious that P_1 and P_2 , multiplying from the left ρ_{12} , give one and the same operator, i. e., that they are twins, and it is also obvious that they both give the same irrespectively if they multiply ρ_{12} from the left or from the right, i. e., that they are strong twin projectors. \square

In view of (5), it is clear that biorthogonal decomposition of a statistical operator can be, in principle, *continued*: If, e. g., ρ'_{12} in a biorthogonal decomposition (6) is, in its turn, decomposed into biorthogonal statistical operators and replaced in (6), then any two of the new terms are biorthogonal etc.

An extreme case of a biorthogonal mixture is a *separable* one:

$$\rho_{12} = \sum_k w_k \left(\rho_1^{(k)} \otimes \rho_2^{(k)} \right), \quad (9)$$

where

$$\forall k : \quad w_k > 0, \quad \rho_i^{(k)} > 0, \quad \text{Tr} \rho_i^{(k)} = 1, \quad i = 1, 2; \quad \sum_k w_k = 1$$

(" $\rho > 0$ " denotes positivity of the operator). This decomposition cannot, of course, always be carried out, but examples are well known. For instance, if one performs ideal measurement of the z-component of spin of the first particle in a singlet two-particle state, one ends up with

$$\rho_{12} \equiv (1/2) \left(|z+\rangle_1 \langle z+|_1 \otimes |z-\rangle_2 \langle z-|_2 + |z-\rangle_1 \langle z-|_1 \otimes |z+\rangle_2 \langle z+|_2 \right).$$

This is obviously a biorthogonal separable mixture.

One wonders if, at the price of relaxing the requirement of statistical-operator terms as slightly as possible, there could exist a *general* decomposition into *uncorrelated* terms (like in (9)).

To find an affirmative answer, we take resort to the known case of general (entangled or disentangled) composite-system *state vectors* and their Schmidt canonical expansions. Let us sum up the relevant information on this [2].

The Schmidt canonical expansion (also called Schmidt biorthogonal expansion) of an arbitrary pure state vector $|\Phi\rangle_{12}$ of a composite system is expressed in terms of *its canonical entities*. They are the following:

(i) *The reduced statistical operators* (subsystem states) ρ_1 ($\equiv \text{Tr}_2 |\Phi\rangle_{12}\langle\Phi|_{12}$) and ρ_2 (defined symmetrically) are well known.

(ii) The spectral forms of the reduced statistical operators are

$$\rho_1 = \sum_i r_i |i\rangle_1\langle i|_1, \quad \rho_2 = \sum_i r_i |i\rangle_2\langle i|_2, \quad \forall i: r_i > 0. \quad (10a, b)$$

(Note that the positive spectra -multiplicities included - are always equal.)

(iii) Finally, the mentioned expansion utilizes the (antiunitary) *correlation operator* U_a , which maps the range $\mathcal{R}(\rho_1)$ onto the range $\mathcal{R}(\rho_2)$. (Note that they are always equally dimensional in the pure state case). The correlation operator is determined by $|\Phi\rangle_{12}$, and, in turn, in conjunction with ρ_1 , it determines $|\Phi\rangle_{12}$.

The *Schmidt canonical expansion* reads:

$$|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes (U_a |i\rangle_1)_2. \quad (11)$$

The normalized characteristic vectors $|i\rangle_2$ in (10b) may (and need not) be chosen to be equal to $(U_a |i\rangle_1)_2$.

3. Hermitian Schmidt canonical expansion of statistical operators

It is well known that linear Hilbert-Schmidt operators A , i. e., those with a finite Hilbert-Schmidt norm $(\text{Tr} A^\dagger A)^{1/2}$, form a Hilbert space in their turn. Writing the operator A as a (Hilbert-Schmidt) supervector $|A\rangle$, the scalar product is

$$\langle A || B \rangle \equiv \text{Tr} A^\dagger B.$$

Since for every statistical operator ρ , one has $\text{Tr} \rho^2 \leq 1$, it is a Hilbert-Schmidt operator. Therefore, *every statistical operator has a Schmidt canonical expansion*.

The trouble is that the operators that take the place of the statistical operators $\rho_i^{(k)}$, $i = 1, 2$ in (9) are, in general, linear operators. This might be a too wide generalization. One wonders if one could be confined to Hermitian operators.

When we view the operators as supervectors, then we must view *adjoining* of operators as *an antiunitary operator* the square of which is the identity operator, i. e., which is an *involution*. Hence, we denote adjoining by $V_1^{(a)} \otimes V_2^{(a)}$ for a composite system. Hermitian are the operators that are *invariant* under the action of this antiunitary involution.

Fortunately, the Schmidt canonical expansion can always be expressed in terms of Hermitian operators. We put this in a more precise and a more detailed way. But it is simpler to return to the Hilbert space of state vectors.

Theorem 2. Let $V_1^{(a)} \otimes V_2^{(a)}$ be a given antiunitary involution acting on composite-system state vectors. One has the equivalence:

$$(V_1^{(a)} \otimes V_2^{(a)}) |\Phi\rangle_{12} = |\Phi\rangle_{12} \quad \Leftrightarrow \quad [\rho_i, V_i^{(a)}] = 0, \quad i = 1, 2; \quad V_2^{(a)} U_a V_1^{(a)} = U_a, \quad (12)$$

where ρ_i , U_a are the above mentioned canonical entities of $|\Phi\rangle_{12}$. (Note that in the last relation we, actually, have the restriction of $V_1^{(a)}$ to $\mathcal{R}(\rho_1)$.)

Proof. Let $|\Phi\rangle_{12}$ be invariant under the action of the antiunitary involution. Then

$$\begin{aligned} V_1^{(a)} \rho_1 V_1^{(a)} &= V_1^{(a)} \left(\text{Tr}_2 |\Phi\rangle_{12} \langle \Phi|_{12} \right) V_1^{(a)} = \\ \text{Tr}_2 \left(V_1^{(a)} |\Phi\rangle_{12} \langle \Phi|_{12} V_1^{(a)} \right) &= \text{Tr}_2 \left[V_1^{(a)} \left(V_1^{(a)} \otimes V_2^{(a)} \right) |\Phi\rangle_{12} \langle \Phi|_{12} \left(V_1^{(a)} \otimes V_2^{(a)} \right) V_1^{(a)} \right] = \\ \text{Tr}_2 \left(V_2^{(a)} |\Phi\rangle_{12} \langle \Phi|_{12} V_2^{(a)} \right) &= \text{Tr}_2 |\Phi\rangle_{12} \langle \Phi|_{12} = \rho_1, \end{aligned}$$

and symmetrically for ρ_2 . One has to note that an antiunitary involution equals its inverse and its adjoint. Further, use has been made of some known basic properties of partial traces (which are analogous to the well known ones for ordinary traces).

Commutation of ρ_1 with $V_1^{(a)}$ allows one to choose the characteristic basis $\{|i\rangle_1 : \forall i\}$ of the former spanning its range consisting of vectors invariant under the action of $V_1^{(a)}$ (cf [4]).

Now, let us take the Schmidt canonical expansion (11) in terms of an invariant basis. Then

$$(V_1^{(a)} \otimes V_2^{(a)}) |\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes V_2^{(a)} (U_a |i\rangle_1)_2.$$

Since $|\Phi\rangle_{12}$ was assumed to be invariant, we have also

$$|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes V_2^{(a)} (U_a |i\rangle_1)_2.$$

The second tensor factor in each term is uniquely determined by the LHS and the corresponding first tensor factor (as a partial scalar product, cf [2]). Comparison with (11) then shows that

$$\forall i : \quad V_2^{(a)} U_a |i\rangle_1 = U_a |i\rangle_1.$$

Since $|i\rangle_1 = V_1^{(a)} |i\rangle_1$, we further have

$$V_2^{(a)} U_a V_1^{(a)} = U_a$$

as claimed.

Conversely, if the main canonical entities are in the relation to the antiunitary involutions as stated in (12), then we can expand $|\Phi\rangle_{12}$ in a characteristic basis of ρ_1 spanning its range that is invariant under the antilinear operator. Then (11) immediately reveals that, as a consequence, $|\Phi\rangle_{12}$ is invariant under $V_1^{(a)} \otimes V_2^{(a)}$. \square

Corollary 1. Every composite-system statistical operator ρ_{12} has a *Hermitian* Schmidt canonical expansion.

Proof. Since every ρ_{12} , being Hermitian, is invariant under the antiunitary involution $V_1^{(a)} \otimes V_2^{(a)}$, Theorem 2 immediately implies that ρ_{12} , upon super vector normalization,

has a Schmidt canonical expansion in terms of Hermitian operators. \square

Returning to a biorthogonal mixture, one wonders if one can *continue* such a decomposition by writing each term in a Hermitian Schmidt canonical expansion in order to obtain the latter expansion for the entire statistical operator. The answer is affirmative on account of the following result.

Going back to (5), we add a fourth equivalent property.

Proposition 1. Two statistical operators ρ' and ρ'' are *orthogonal* if and only if they are *orthogonal as Hilbert-Schmidt supervectors*.

Proof. It is obvious that orthogonality (in the sense of (5)) implies Hilbert-Schmidt orthogonality. To see the converse implication, we make use of the fact that every statistical operator has a purely discrete spectrum [3], and we decompose the statistical operators in terms of characteristic vectors corresponding to positive characteristic values:

$$\begin{aligned} \langle \rho' || \rho'' \rangle &= \text{Tr} \rho' \rho'' = \text{Tr} \sum_k r_k |k\rangle \langle k| \sum_j \bar{r}_j |j\rangle \langle j| = \\ &= \sum_k \sum_j r_k \bar{r}_j |\langle j || k \rangle|^2. \end{aligned}$$

Hence,

$$\langle \rho' || \rho'' \rangle = 0 \quad \Rightarrow \quad \rho' \rho'' = 0$$

(cf the third relation in (5)). \square

If (A_1, A_2) is a pair of *twin observables*, then, as it was stated (cf also [1]), the detectable parts A'_i , $i = 1, 2$, have a common purely discrete spectrum $\{a_n : \forall n\}$ (with, in general, different multiplicities), and the corresponding characteristic projectors $\{P_i^{(n)} : i = 1, 2 \ \forall n\}$, are also pairs of twins.

Definition 3. If *all* mentioned characteristic projector pairs $(P_1^{(n)}, P_2^{(n)})$ are strong twin projectors, then (A_1, A_2) is a pair of *strong twin observables*. If some of the detectable characteristic twin projectors are strong and some weak, we say that we have *partially strong* (or, synonymously, *partially weak*) twin observables. If all the mentioned twin projectors are weak, then we have a *weak* pair of twin observables.

Evidently, a pair (A_1, A_2) of nontrivial twin observables for ρ_{12} is a pair of strong ones if and only if

$$[A_i, \rho_{12}] = 0, \quad i = 1, 2 \tag{13}$$

is valid. This is so because commutation with all characteristic projectors is equivalent to commutation with the Hermitian operator itself.

Strong twin observables, by means of their strong characteristic twin projectors, lead to a generalization of (3):

$$\rho_{12} = \sum_n P_1^{(n)} \rho_{12} = \sum_n w_n \rho_{12}^{(n)}, \quad (14a)$$

where

$$\forall n : \quad w_n \equiv \text{Tr} \rho_{12} P_1^{(n)}, \quad \rho_{12}^{(n)} \equiv (w_n)^{-1} P_1^{(n)} \rho_{12}, \quad (14b)$$

and any two terms in (14a) are biorthogonal. (Note that we utilize the entire characteristic projectors, which are the orthogonal sums $P_1^{(n)} = (P_1')^{(n)} \oplus (P_1'')^{(n)}$ parallelling $\mathcal{H}_1 = \mathcal{R}(\rho_1) \oplus \mathcal{R}^\perp(\rho_1)$ because $(P_1')^{(n)} \rho_{12} = P_1^{(n)} \rho_{12}$.)

Proposition 2. If

$$\rho_1^{(n)} \equiv \text{Tr}_2 \rho_{12}^{(n)},$$

and symmetrically for $\rho_2^{(n)}$, are the reduced statistical operators of the terms in the biorthogonal mixture (14a), then

$$P_i^{(n)} \rho_i^{(n)} = \rho_i^{(n)}, \quad i = 1, 2, \quad (15a)$$

or equivalently,

$$\mathcal{R}(\rho_i^{(n)}) \subseteq \mathcal{R}(P_i^{(n)}), \quad i = 1, 2. \quad (15b)$$

Proof. On account of the definition of (14a), one has $P_i^{(n)} \rho_{12}^{(n)} = \rho_{12}^{(n)}$. Taking the opposite-subsystem partial trace, one obtains $P_i^{(n)} \rho_i^{(n)} = \rho_i^{(n)}$ $i = 1, 2$. \square

Corollary 2. If the detectable part A_1' of a twin observable A_1 has a *nondegenerate* characteristic value a_n corresponding to a strong characteristic twin projector $(P_1')^{(n)} = |\psi^{(n)}\rangle_1 \langle \psi^{(n)}|_1$, $|\psi^{(n)}\rangle_1 \in \mathcal{R}(\rho_1)$, then the term in the biorthogonal mixture (14a) that corresponds to it has the form

$$w_n |\psi^{(n)}\rangle_1 \langle \psi^{(n)}|_1 \otimes \rho_2^{(n)}, \quad (16)$$

where $\rho_2^{(n)}$ is a (second-subsystem) state and (16) is a term in a final Hermitian Schmidt canonical expansion of ρ_{12} .

Any biorthogonal decomposition of a composite-system statistical operator ρ_{12} (into two or more terms) can be continued in each term separately into a Schmidt canonical expansion of ρ_{12} in terms of Hermitian operators.

The biorthogonal decomposition is *an intermediate step*. This is similar to the case when we can partially diagonalize the Hamiltonian of a quantum system (due to some symmetry e. g.). The diagonalization is then continued separately with each submatrix on the diagonal of the Hamiltonian.

The continuation from a biorthogonal mixture to a Hermitian Schmidt canonical expansion can always be performed, in principle, "by brute force": diagonalizing the

reduced statistical superoperator $\hat{\rho}_1$ of the normalized supervector $|\rho_{12}\rangle$ (analogously as it is done for an ordinary state vector), and by finding an invariant basis for $V_1^{(a)}$ in each characteristic subspace thus obtained [4].

The Hermitian Schmidt canonical expansion of a composite-system statistical operator will, hopefully, find numerous applications in quantum communication and information theory because it lies at the basis of entanglement. One of the applications is evaluating all the twin observables. This is illustrated elsewhere [5].

4. Weak twins and nonhermitian Schmidt canonical expansion

For the sake of completeness it is desirable to investigate decomposition (3) also for a weak nontrivial twin projector P_1 . First, we take an analytical view of Theorem 1 to realize that the biorthogonality of the two terms in (3) is connected with the twin property (strong or weak), and the strong twin property corresponds to the hermiticity of the terms. Let us put this more precisely.

Definition 4. A decomposition

$$\rho_{12} = A_{12} + B_{12}$$

of a composite-system statistical operator ρ_{12} into two linear operators is *biorthogonal* if there exist two opposite-subsystem projectors (P_1, P_2) such that

$$\begin{aligned} A_{12} &= P_1 A_{12} = P_2 A_{12}, & 0 &= P_1 B_{12} = P_2 B_{12}; \\ 0 &= P_1^\perp A_{12} = P_2^\perp A_{12}, & B_{12} &= P_1^\perp B_{12} = P_2^\perp B_{12}. \end{aligned}$$

It is clear from Theorem 1 that any biorthogonal mixture (of states) (6) satisfies the generalized definition of biorthogonality given in Definition 4. Having in mind (3), it is also evident that biorthogonality is equivalent to the existence of a pair of twin projectors (weak or strong). Finally, the strongness property of the twins is equivalent to the hermiticity of the terms in (3), which results in having statistical operator terms (and a mixture).

Theorem 3. If (P_1, P_2) is a pair of *weak twin projectors* for a composite-system statistical operator ρ_{12} , then the terms in (3) are super vectors, and replacing each by a (nonhermitian) Schmidt canonical expansion, one obtains an expansion of the same kind for the entire statistical operator.

Proof. Since in

$$1 \geq \text{Tr} \rho_{12}^2 = \text{Tr} \rho_{12} P_1 \rho_{12} + \text{Tr} \rho_{12} P_1^\perp \rho_{12}$$

the terms are nonnegative (as traces of positive operators), the terms in (3) are Hilbert-Schmidt operators, i. e., super vectors. Suppose we have expanded the first term in (3) in the Schmidt canonical way:

$$P_1 \rho_{12} = c \sum_i r_i^{1/2} A_1^{(i)} \otimes B_2^{(i)},$$

where c is a normalization constant (because the statistical operator is not a super state vector unless it is a pure state). Since the LHS is invariant under P_1 , so is each first-subsystem linear operator $A_1^{(i)}$, because the second factors in the expansion have unique corresponding first factors. If we expand also the second term in (3) in the Schmidt canonical way

$$P_1^\perp \rho_{12} = c' \sum_j r_j^{1/2} C_1^{(j)} \otimes D_2^{(j)},$$

then, analogously, invariance of each factor $C_1^{(j)}$ under P_1^\perp follows. This results in super vector orthogonality:

$$\forall i, j : \quad \text{Tr}[(A_1^{(i)})^\dagger C_1^{(j)}] = \text{Tr}[(A_1^{(i)})^\dagger P_1 P_1^\perp C_1^{(j)}] = 0.$$

The symmetrical argument goes for the second factors and P_2 . Thus, replacing both terms in (3) by their nonhermitian Schmidt canonical expansions, we have biorthogonality between any term of the first expansion and any term of the second one. Therefore, we have an expansion of the same kind for ρ_{12} . \square

It is now clear that also in the case of weak twin projectors the decomposition (3) can be *continued*, but this time *to a nonhermitian Schmidt canonical expansion*.

As it was stated, I expect that Hermitian Schmidt canonical expansion of composite-system statistical operators, and biorthogonal mixtures that lead to it, will soon find important application in quantum communication and quantum information theory. But, maybe, also the nonhermitian version will be useful.

After all, a nonhermitian expansion need not be wild and far fetched from the physical point of view. Let me illustrate this by the obvious fact that a Schmidt canonical expansion of a state vector $|\Phi\rangle_{12}$

$$|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 |i\rangle_2, \quad \forall i \neq i' : \quad \langle i|_p |i'\rangle_p = 0, \quad p = 1, 2$$

immediately results in a nonhermitian Schmidt canonical expansion of the statistical operator $|\Phi\rangle_{12}\langle\Phi|_{12}$:

$$|\Phi\rangle_{12}\langle\Phi|_{12} = \sum_i \sum_{i'} r_i^{1/2} r_{i'}^{1/2} |i\rangle_1 \langle i'|_1 \otimes |i\rangle_2 \langle i'|_2.$$

Finally, let us return to separable mixtures.

5. Nontrivial twin projectors for separable mixtures

Let (9) be a general separable mixture. Let us clarify under what conditions it has nontrivial twin events.

Theorem 4. A general separable mixture (9) has a nontrivial twin projector P_1 if and only if the set of all values of the index "k" is the union of two nonoverlapping subsets,

say, consisting of "k'" values and of "k'" values respectively, and, when (9) is rewritten accordingly:

$$\rho_{12} = \sum_{k'} w_{k'} \rho_1^{(k')} \otimes \rho_2^{(k')} + \sum_{k''} w_{k''} \rho_1^{(k'')} \otimes \rho_2^{(k'')}, \quad (17a)$$

then one has biorthogonality between the two groups of terms:

$$\forall k', \forall k'' : \quad \rho_i^{(k')} \rho_i^{(k'')} = 0, \quad i = 1, 2. \quad (17b)$$

Before we prove the theorem, we first prove subsidiary results.

Lemma 1. Let

$$\rho_{12} = \sum_m w_m |\Psi^{(m)}\rangle_{12} \langle \Psi^{(m)}|_{12}$$

be an arbitrary pure-state mixture. Then, a pair of subsystem observables (A_1, A_2) are twins for ρ_{12} *if and only if* they are twins for *all* pure term-states.

Proof. *Necessity* follows from the general result that all twins of ρ_{12} are also twins of all state vectors from the topological closure $\bar{\mathcal{R}}(\rho_{12})$ of the range of ρ_{12} (cf section 3, C1 in [1]). As well known, the vectors $\{|\Psi^{(m)}\rangle_{12} : \forall m\}$ span the mentioned subspace.

Sufficiency is obvious. \square

Lemma 2. Let

$$\rho_{12} = \sum_k w_k \rho_{12}^{(k)}$$

be an arbitrary mixture. The pair (A_1, A_2) are twin observables for ρ_{12} *if and only if* they are twin observables for *all* term states $\rho_{12}^{(k)}$.

Proof is immediately obtained from Lemma 1 if one rewrites each term state as a pure-state mixture. \square

Lemma 3. An *uncorrelated state* $\rho_1 \otimes \rho_2$ has only trivial twins.

Proof is an immediate consequence of the fact that the tensor factors of a nonzero uncorrelated vector, say $a \otimes b$, are unique up to an arbitrary nonzero complex number α , but if a is replaced by αa , b must be replaced by $(1/\alpha)b$.

Applying this to supervectors in case of twins, we have

$$A_1 \rho_1 \otimes \rho_2 = \rho_1 \otimes A_2 \rho_2,$$

if $A_1 \rho_1 = \alpha \rho_1$, then $\rho_2 = (1/\alpha) A_2 \rho_2$. \square

Proof of Theorem 3 now immediately follows from Lemma 2 and Lemma 3. Namely, the two groups of terms stated in the Theorem, make up the two terms in (3). \square

Corollary 3. Nontrivial twin events of a separable mixture (9) are *necessarily strong twin events*.

Proof is obvious if one applies Lemmas 2 and 3 and if adjoining is made use of. \square

Corollary 4. If (A_1, A_2) are nontrivial twin observables for a separable mixture (9), they are strong twin observables (cf Definition 3), and the mixture terms can be grouped into as many biorthogonal groups of terms as there are distinct characteristic values of A_1 in $\mathcal{R}(\rho_1)$ (generalization of (17a,b)).

It is known that if a statistical operator and a Hermitian operator commute, then the corresponding state can be written as a mixture so that each term-state has a definite value of the corresponding observable [6]. But, for the same statistical operator, there are also mixtures violating this.

To take an example, let us think of an unpolarized mixture of spin-one-half states: $\rho = (1/2)I$ (in the two-dimensional spin factor space). This statistical operator commutes with s_z , nevertheless one can write down the mixture

$$\rho = (1/2) \left(|x, +\rangle\langle x, +| + |x, -\rangle\langle x, -| \right) = (1/2)I,$$

in which the term-states do not have a definite value of the z-component.

It is interesting that in the case of a separable mixture with a nontrivial twin observable it is necessarily its term-states that have the sharp detectable values of the corresponding observable.

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